Screening Two Types

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Here: screening two types

- Impose only quasilinearity

Results

A general characterization of optimal mechanisms

Two applications

1. Bundling
2. Vertical and horizontal differentiation

Model

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Two types $\{t_1, t_2\}$, probabilities $1 - q, q$

A set of "alternatives" A

Value $v(t, a)$, $v(t, 0) = 0$

- Payoff $v(t, a) - p$

Cost $c(a)$ normalized to zero

Goal: profit-maximizing IC&IR mechanisms

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- Today: allow for randomization. $(x, p) : \{t_1, t_2\} \to \Delta(A) \times R$

Application 1: Bundling

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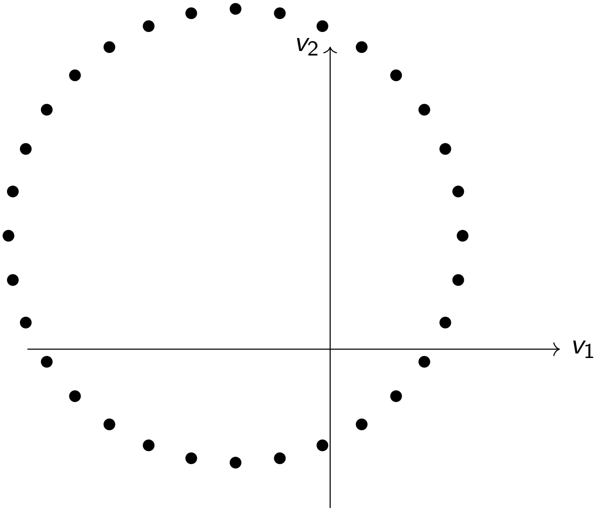
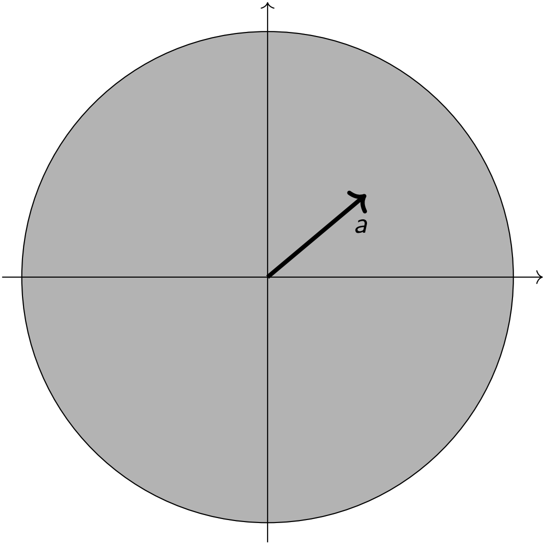


Application 2: Vertical and Horizontal differentiation

$A =$ all points within the circle.



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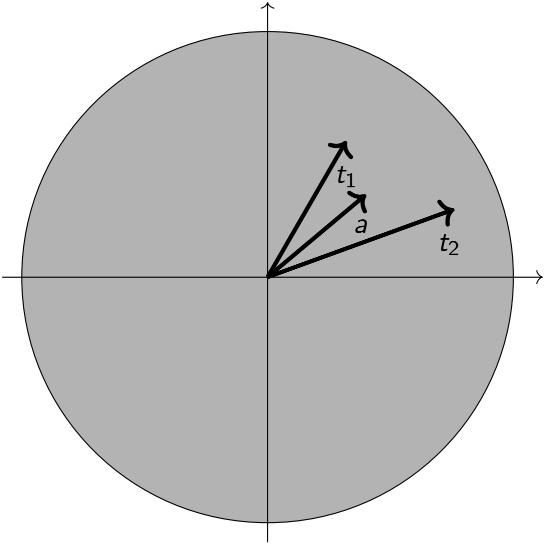

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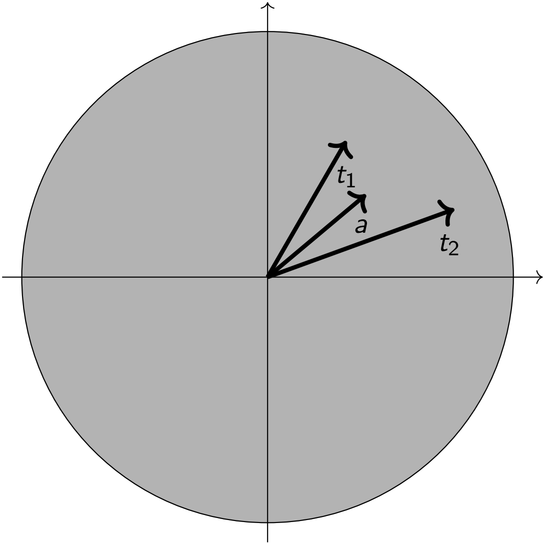
$A =$ all points within the circle. $v(t, a) = t \cdot a$.



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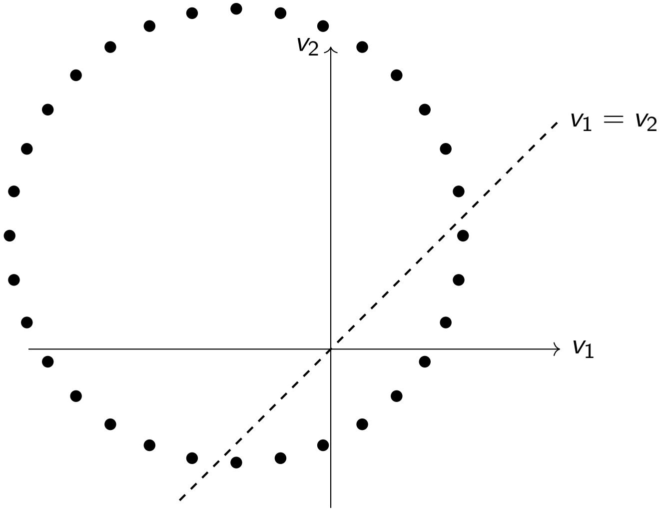

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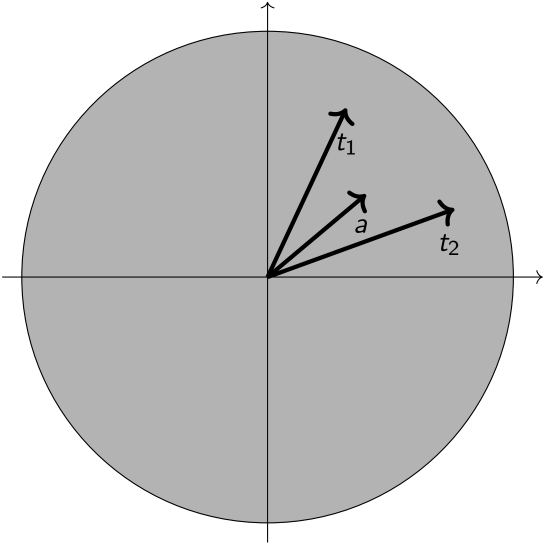

Back to General Model

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A set of "alternatives" A

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First-best is feasible (is IC) if

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Proposition

If (1) $\Rightarrow$ First-best mechanism is feasible and therefore optimal.
If not (1) $\Rightarrow$ see next slide.

Result continued

$$0 \geq v(t_1, \bar{a}(t_2)) - v(t_2, \bar{a}(t_2)); 0 \geq v(t_2, \bar{a}(t_1)) - v(t_1, \bar{a}(t_1))$$

Proposition (continued)

Suppose $v(t_2, \bar{a}(t_1)) > v(t_1, \bar{a}(t_1))$.

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1. *t_2 is "the high type":*
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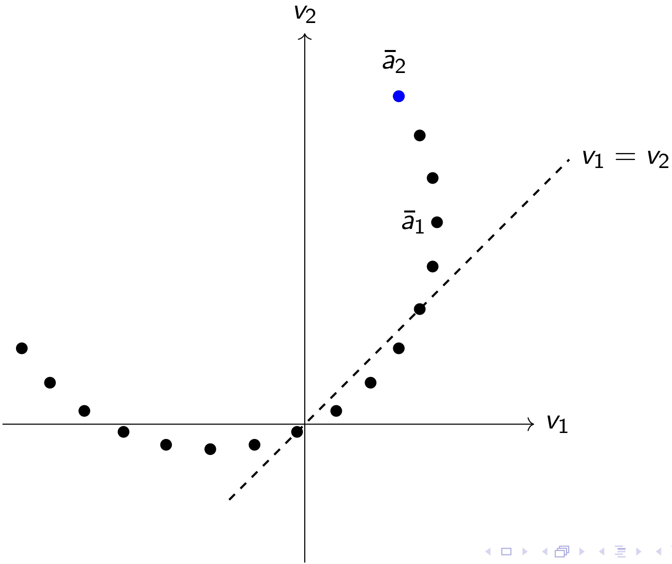
2. *t_1 is "the low type":*
   c. *Its IR binds (pins down payment given t_1's allocation)*
   d. *Allocation $\in \arg\max v(t_1, a) - qv(t_2, a)$, s.t. $v(t_2, a) \geq v(t_1, a)$*

Allocation of t_1 when $v(t_2, \bar{a}(t_1)) > v(t_1, \bar{a}(t_1))$



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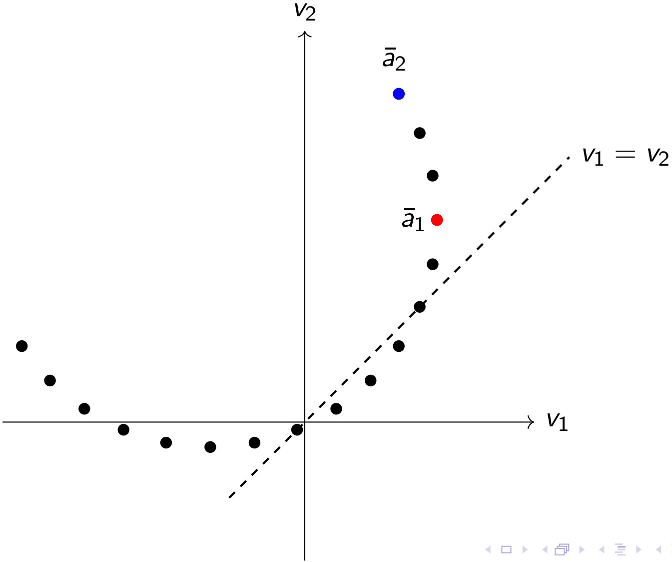
$$\text{allocation} = \arg\max v_1 - qv_2 \ (\text{s.t.}, \ v_2 \geq v_1)$$



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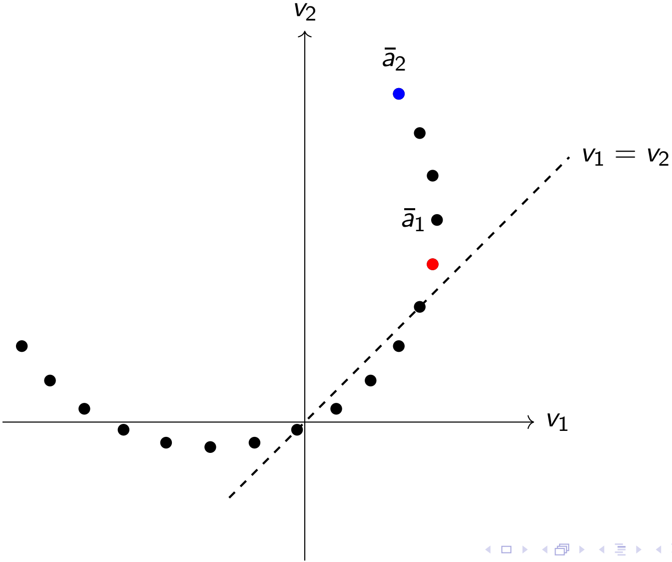

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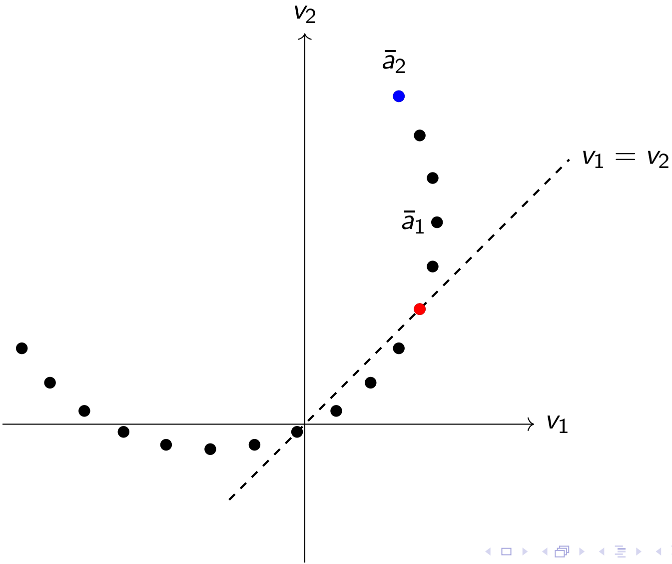
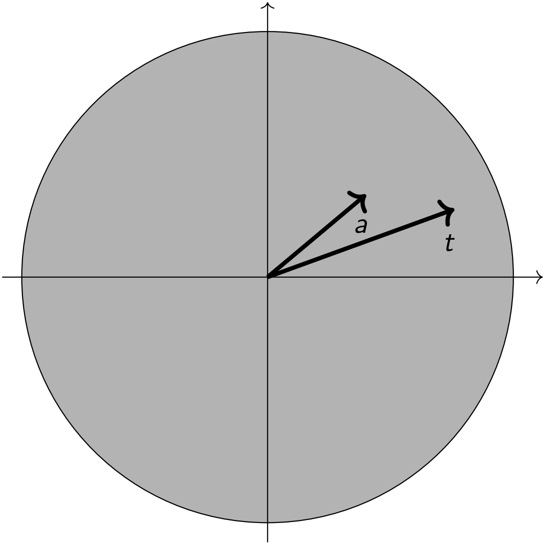


Vertical + Horizontal differentiation

$c(a) = c \cdot s(a)$, $v(t, a) = a \cdot t$



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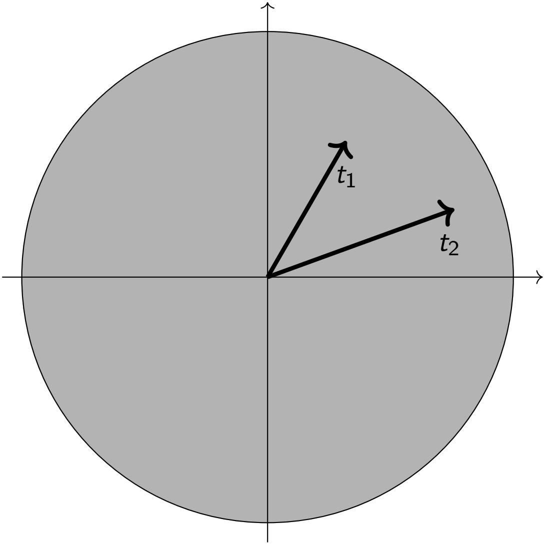

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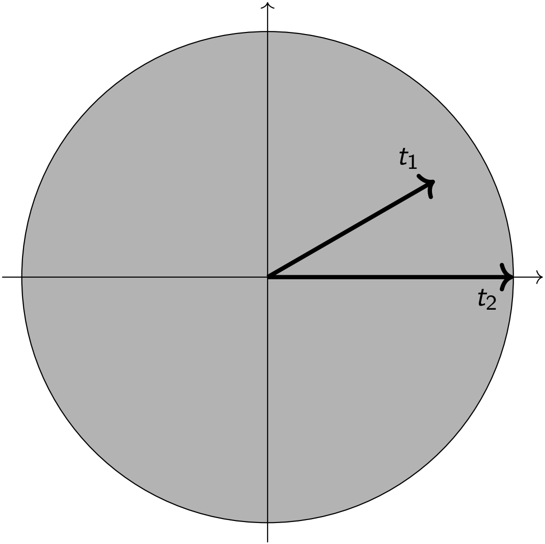

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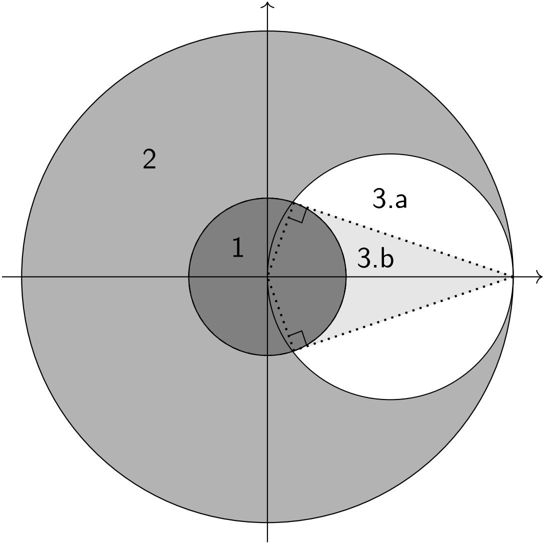

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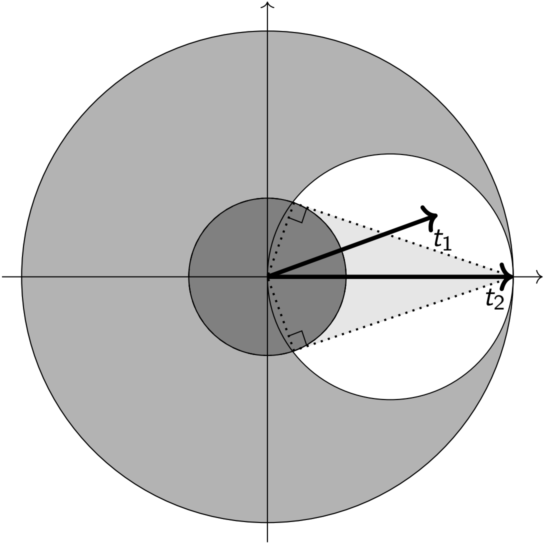

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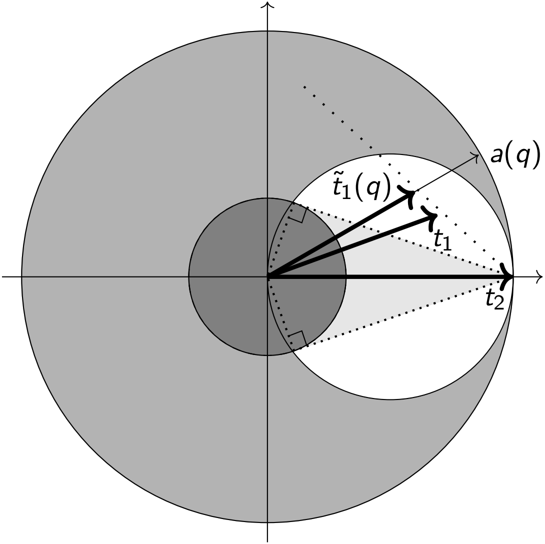

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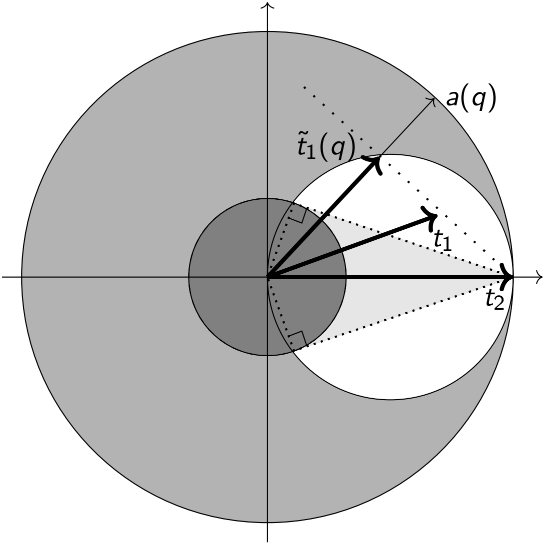

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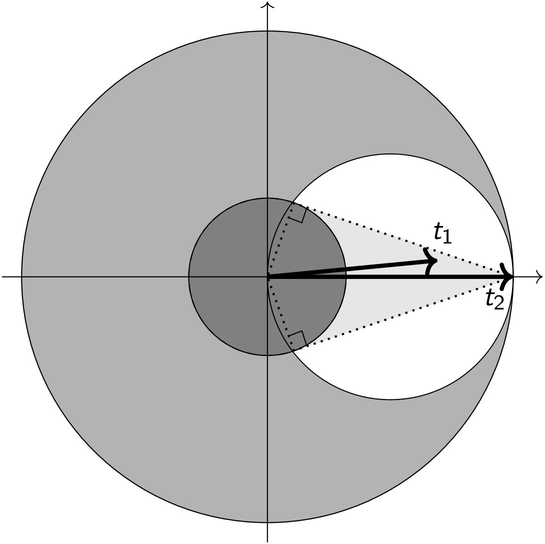

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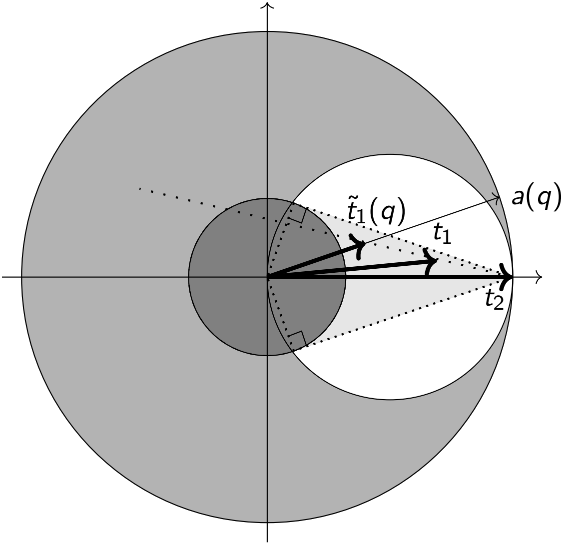

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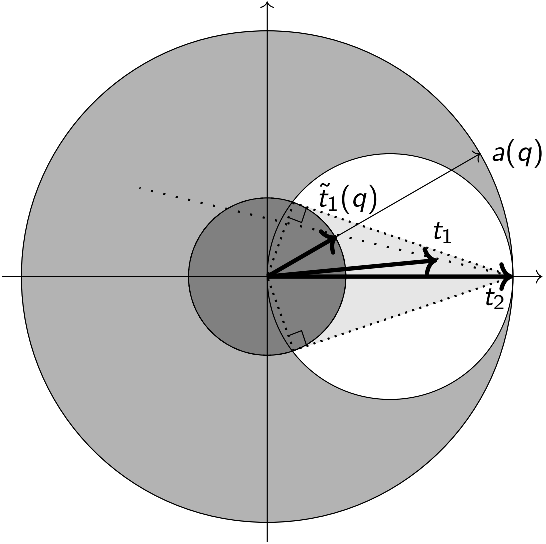

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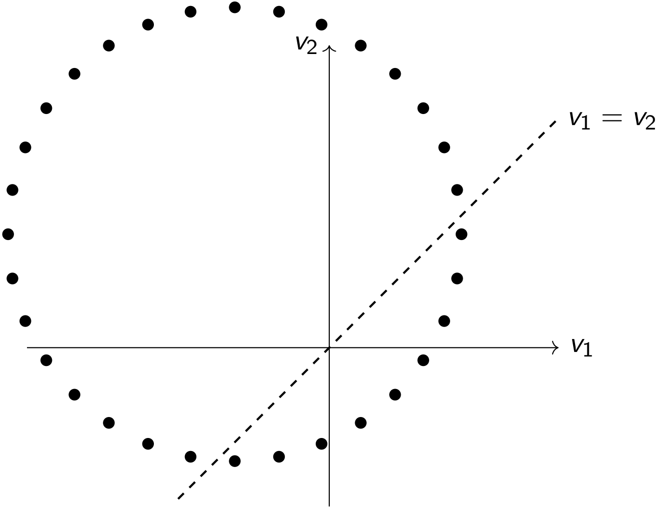

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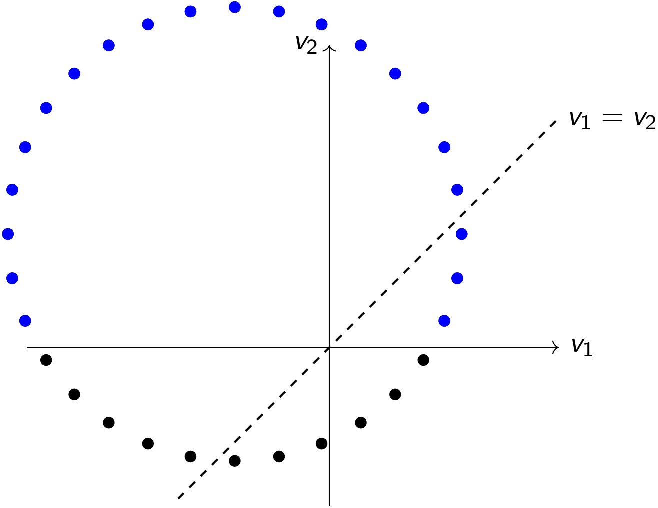

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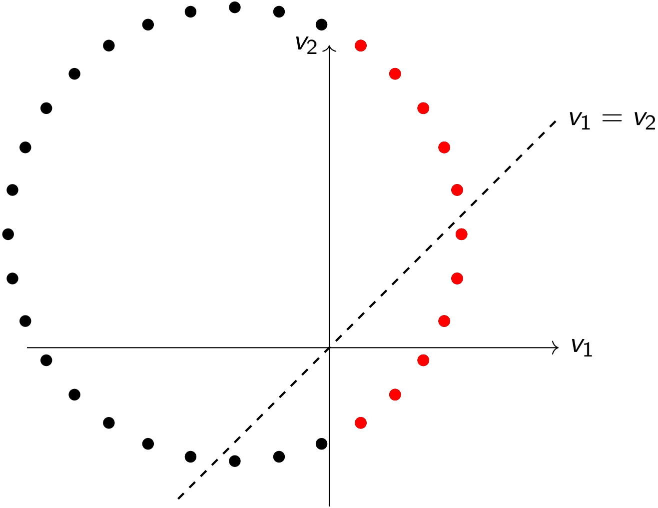

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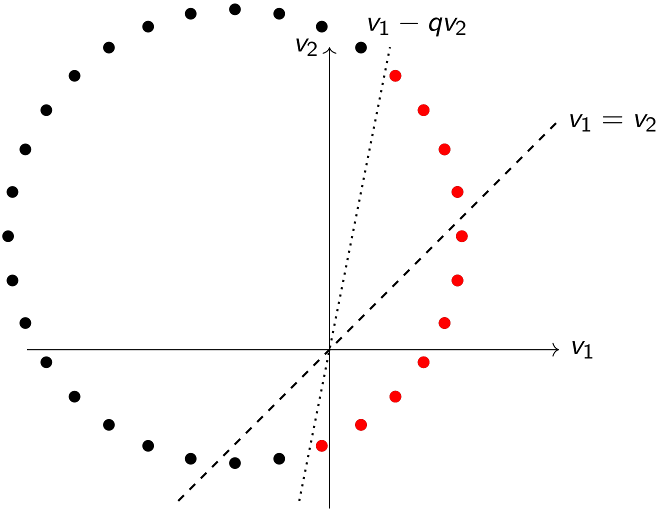

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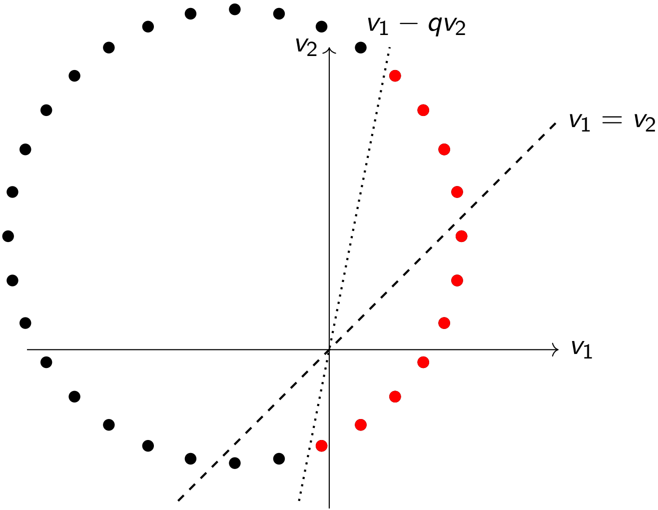

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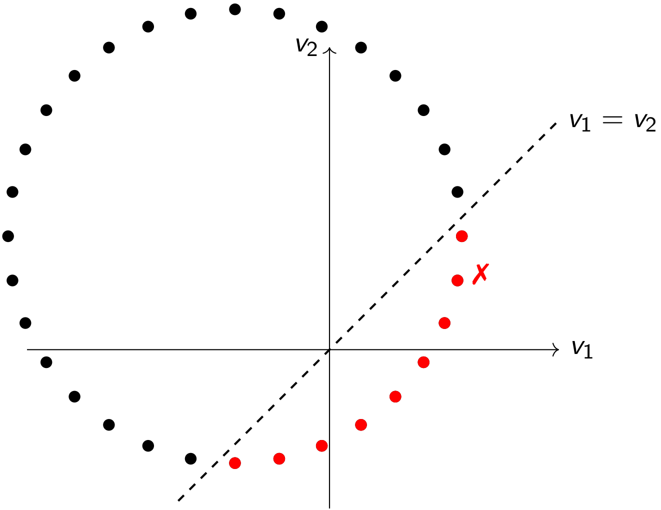

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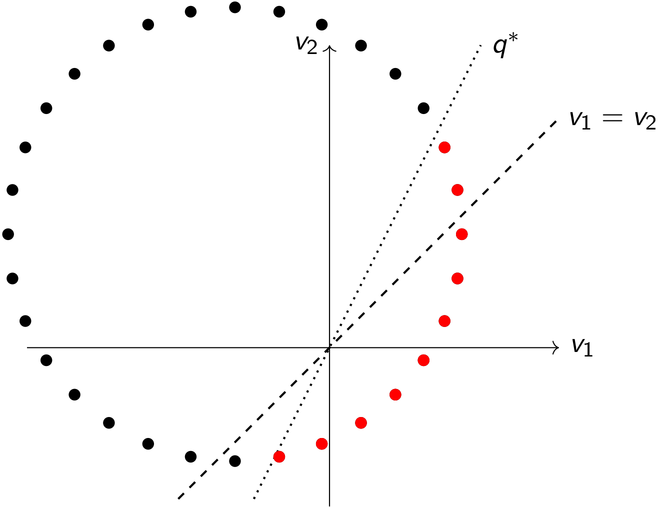

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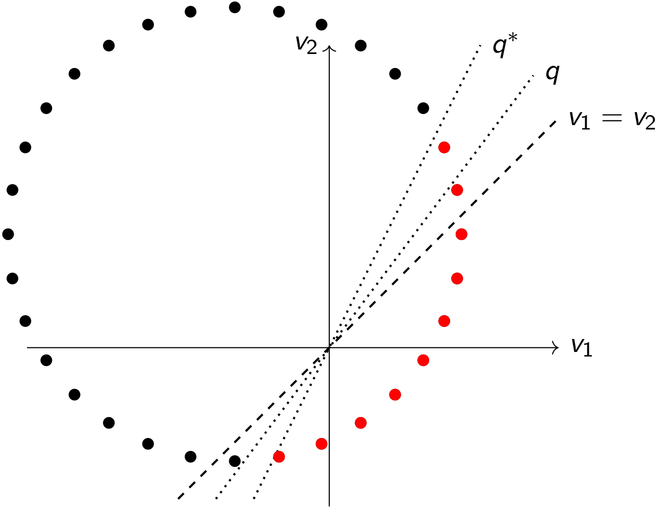

A general characterization of optimal mechanisms with two types

- A simple comparison specifies which type is high and which is low

Two applications

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